

# Covering Graphs by Colored Stable Sets

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## Abstract

Let  $G = (V, R \cup B)$  be a multigraph with red and blue edges.  $G$  is an  $R/B$ -split graph if  $V$  is the union of a red and a blue stable set.  $R/B$ -split graphs yield a common generalization of split graphs and König graphs. It is shown, for example, that  $R/B$ -split graphs can be recognized in polynomial time. On the other hand, finding a maximal  $R/B$ -subgraph is  $\mathcal{NP}$ -hard already for the class of comparability graphs of series-parallel orders. Moreover, there can be no approximation ratio better than  $31/32$  unless  $\mathcal{P}=\mathcal{NP}$ .

## 1 Introduction

A subset  $S$  of the node set  $V$  of a given graph  $G = (V, E)$  is said to be *stable* if no pair of nodes in  $S$  is joined by an edge in  $E$ . The problem to determine a stable set of maximal cardinality is known to be  $\mathcal{NP}$ -hard for general graphs. But for a large class of graphs, *e.g.* perfect graphs, the MAX STABLE SET PROBLEM has been shown to be polynomially solvable ([2], ch. 67).

In the present article, we consider multigraphs  $G = (V, E)$  whose sets of edges consist of "red" and "blue" edges, say  $E = R \cup B$ . We are interested in covering the node set  $V$  of such a graph  $G$  by a red and a blue stable set to the best possible, *i.e.* we want to maximize the cardinality of the union of a red and a blue stable set, where a *red stable set* denotes a stable set in the *red graph*  $G_R = (V, R)$  and a *blue stable set* refers to a stable set in the *blue graph*  $G_B = (V, B)$ .

If a graph  $G = (V, R \cup B)$  has the property that the whole vertex set  $V$  can be covered by a red and a blue stable set we will call  $G$  an  $R/B$ -split graph as  $V$  can be split into a red and a blue stable set.

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Section 2 shows that one can decide in polynomial time whether or not a given graph is an  $R/B$ -split graph. It turns out that the model of  $R/B$ -split graphs provides a natural common generalization of classical split graphs (see Földes and Hammer [6]) and graphs with the König Property (see Lovász and Plummer [7], p. 222). In our terminology, a (classical) *split graph* is a one-colored graph whose node set can be split into a stable set and a clique, while a graph with the *König Property* is a graph in which the size of a maximal matching equals the size of a minimal node cover.

If  $G$  is not an  $R/B$ -split graph one might want to determine a maximal subgraph of  $G$  that is an  $R/B$ -split graph, *i.e.* one wishes to cover as many nodes as possible by the union of a red and a blue stable set. This optimization problem is easily seen to be  $\mathcal{NP}$ -hard in general as it already reduces to the original MAX STABLE SET PROBLEM if  $R = \emptyset$  or  $B = \emptyset$ . Section 3 discusses some polynomially solvable instances of the problem. If, for example,  $G_R = G_B$  is the comparability graph of a partial order, the problem amounts to determining a maximal subset that can be expressed as the union of two antichains in the partial order. This maximization problem is well-known to be solvable in polynomial time even for the union of  $k$  antichains (see, *e.g.*, Frank [1]).

Interestingly, the problem of determining a maximal union of a red and a blue antichain turns out to be  $\mathcal{NP}$ -hard already for the class of series-parallel orders as we show in Section 4.

## 2 Red/Blue-Split Graphs

Given a graph  $G = (V, R \cup B)$  with sets  $R$  and  $B$  of red, resp. blue, edges, we want to decide whether there exists a partition  $V = S_R \cup S_B$  of  $V$  into a stable set  $S_R$  in the red graph  $G_R = (V, R)$  and a stable set  $S_B$  in the blue graph  $G_B = (V, B)$ . Note that  $G$  is a multigraph as two nodes may be linked by a red and by a blue edge. We call this decision problem the  $R/B$ -SPLIT PROBLEM and show that it is efficiently solvable.

**Theorem 2.1.** *The  $R/B$ -SPLIT PROBLEM can be efficiently reduced to a 2-SATISFIABILITY PROBLEM.*

*Proof.* The  $R/B$ -SPLIT PROBLEM is equivalent to the (polynomially solvable) problem of either determining a vector  $(x, y) \in \{0, 1\}^{2n}$  that satisfies the 2-SAT formula

$$\bigwedge_{(i,j) \in R} (\neg x_i \vee \neg x_j) \wedge \bigwedge_{(i,j) \in B} (\neg y_i \vee \neg y_j) \wedge \bigwedge_{i \in V} (x_i \vee y_i) \quad (1)$$

or proving that no such vector exists, as we now show.

If a vector  $(\hat{x}, \hat{y}) \in \{0, 1\}^{2n}$  is to satisfy formula (1), each clause must be satisfied individually. Clauses of the form  $\neg x_i \vee \neg x_j$  (resp.  $\neg y_i \vee \neg y_j$ ) guarantee for each pair of nodes  $i, j \in V$  that not both  $i$  and  $j$  will be in

$$S_R := \text{supp}(\hat{x}) \quad (\text{resp. } S_B := \text{supp}(\hat{y}))$$

whenever  $i$  and  $j$  are joined by an edge in the red (resp. blue) graph. Therefore,  $S_R$  will be a red and  $S_B$  a blue stable set. Moreover, each pair of clauses of the form  $x_i \vee y_i$  guarantees each node  $i \in V$  to lie in  $S_R$  or  $S_B$  (or possibly in both sets). So  $V = S_R \cup (S_B \setminus S_R)$  yields a partition of  $V$  into a red and a blue stable set.

On the other hand, if  $V$  can be partitioned into a red stable set  $S_R$  and a blue stable set  $S_B$  the characteristic (support) vectors of  $S_R$  and  $S_B$  will satisfy the formula above.  $\square$

Theorem 2.1 exhibits the R/B-SPLIT PROBLEM to be not more difficult than the 2-SAT PROBLEM. In fact, the two problems are equivalent as any 2-SAT PROBLEM can be solved by solving a corresponding R/B-SPLIT PROBLEM:

**Theorem 2.2.** *The 2-SAT PROBLEM can be efficiently reduced to an R/B-SPLIT PROBLEM.*

*Proof.* Consider an arbitrary 2-SAT instance with  $k$  clauses on the  $n$  variables  $x_i$ ,

$$f(x) = \bigwedge_{j=1}^k (\alpha_j \vee \beta_j),$$

i.e.  $\alpha_j, \beta_j \in \{x_1, \dots, x_n, \neg x_1, \dots, \neg x_n\}$  for all  $j \in \{1, \dots, k\}$ .

Now construct the graph  $G_f$  with red and blue edges on the vertex set

$$V_f := \{x_1, \dots, x_n, \neg x_1, \dots, \neg x_n\},$$

where each vertex  $x_i$  is joined with its complement  $\neg x_i$  by both a red and a blue edge. Furthermore, join a pair of vertices corresponding to  $\neg \alpha_j$  and  $\neg \beta_j$  by a red edge whenever  $\alpha_j \vee \beta_j$  forms a clause in  $f$ .

Observe that  $S_R$  and  $V_f \setminus S_R$  are stable in the red graph  $G'_R$  (resp. blue graph  $G'_B$ ) of  $G_f$  if and only if the elements in  $S_R$  correspond to the true literals in a satisfying variable assignment.  $\square$

One may wonder if the generalized R/B/G-SPLIT PROBLEM of splitting a graph with red, blue and green edges into stable sets is also polynomial. We note

**Lemma 2.3.** *The generalized R/B/G-SPLIT PROBLEM is  $\mathcal{NP}$ -complete.*

*Proof.* The special case  $G_R = G_B = G_G$  of the generalized R/B/G-SPLIT PROBLEM is the well-known  $\mathcal{NP}$ -complete 3-COLORING PROBLEM.  $\square$

Note that the 3-COLORING PROBLEM is polynomial relative to the class of comparability graphs: it is the problem of deciding whether a partially ordered set can be covered by three antichains. To solve this problem one simply calculates a longest chain in the partial order and checks if it has not more than three elements (see, e.g., Thm. 14.1 in [2]). It is an interesting open problem to determine the complexity status of the R/B/G-SPLIT PROBLEM relative to the class of comparability graphs.

We now turn to the discussion of some special instances of the R/B-SPLIT PROBLEM: the case  $R = \bar{B}$ , where the red edge set is the complement of the blue edge set, then the case of the red edges forming a perfect matching in the blue graph and finally the case where the red graph consists of disjoint cliques.

## 2.1 Split Graphs

Recall that a subset  $C \subseteq V$  is a *clique* in the graph  $G = (V, E)$  if every pair of nodes is joined by an edge. So each clique in  $G$  is a stable set in the complement graph  $\bar{G}$  and vice versa.

If the red graph  $G_R$  of  $G = (V, R \cup B)$  is the complement of the blue graph  $G_B$ , we therefore find that  $G$  is an  $R/B$ -split graph if and only if the blue graph  $G_B$  itself can be split into a clique and a stable set, *i.e.* if and only if  $G_B$  is a *split graph*. Földes and Hammer [6] prove that a graph  $G$  is a split graph if and only if  $G$  contains no subgraph isomorphic to  $2K_2, C_4$  or  $C_5$ .

## 2.2 König Graphs

A subset of edges  $M \subseteq E$  of a given graph  $G = (V, E)$  is a *matching* if  $M$  contains no pair of adjacent edges. A matching of size  $|V|/2$  is said to be *perfect*. A subset of nodes  $U \subseteq V$  is called a *node cover* if each edge in  $E$  has at least one endpoint in  $U$ .

Obviously, the size  $\nu(G)$  of a maximum matching can never exceed the size  $\tau(G)$  of a minimum node cover in  $G$ . König [3] establishes the equality

$$\nu(G) = \tau(G) \quad (2)$$

for bipartite graphs. The example of odd circuits shows that this equality is not always true. A graph  $G$  is said to have the *König Property* [7] (or to be a *König graph* for short) if the equality (2) is satisfied.

In order to check whether or not  $G$  is a König graph we need a maximum matching in  $G$  and the *Gallai-Edmonds decomposition* of  $V$  into the set  $D(G)$  of all nodes not covered by at least one maximum matching, the set  $A(G)$  of all neighbors of  $D(G)$  in  $V \setminus D(G)$  and the set  $C(G)$  of the remaining nodes. A maximum matching and the Gallai-Edmonds decomposition

$$V = D(G) \cup A(G) \cup C(G)$$

can be calculated efficiently (for example, with Edmonds' cardinality matching algorithm [7]). The *Gallai-Edmonds structure theorem* [8] states that every maximum matching  $M$  contains a perfect matching of the subgraph  $H$  induced by  $C(G)$  and matches all nodes of  $A(G)$  with nodes of  $D(G)$ .

We can now decide whether  $G$  is a König graph by solving an  $R/B$ -split problem:

**Theorem 2.4.** *Let  $M$  be any maximum matching in  $G = (V, E)$ . Then  $G$  is a König graph if and only if  $D(G)$  is stable in  $G$  and the 2-colored graph  $G' = (V, R \cup B)$ , where  $G'_R = G$  and  $G'_B = (V, M)$ , is an  $R/B$ -split graph.*

*Proof.* Assume first that  $U$  is a node cover with  $|U| = |M|$ . Then for any maximum matching  $M'$  of  $G$ , each matching edge has exactly one endpoint in  $U$ . By the definition of the Gallai-Edmonds decomposition it follows that  $U$  must be contained in  $A(G) \cup C(G)$  implying that  $D(G)$  is a stable set in  $G$ . Moreover,  $V \setminus U$  is stable in the red graph  $G'_R = G$  (because  $U$  is a node cover) and  $U$  is stable in the blue graph  $G'_B$  (because each edge of  $M$  has exactly one endpoint in  $U$ ).

To prove the converse implication, let  $V$  be partitioned into a stable set  $S_R$  in the red graph  $G'_R$  and a stable set  $S_B$  in the blue graph  $G'_B$ . As  $D(G)$  is stable in  $G = G'_R$  we may assume that  $D(G) \subseteq S_R$ , which implies  $S_B \subseteq A(G) \cup C(G)$ , *i.e.* every node in  $S_B$  is incident with exactly one edge of  $M$ . Furthermore, since  $S_R$  is stable in  $G$ , the complement  $S_B$  is a node cover of  $G$ . Therefore  $S_B$  is a node cover of the same size as the maximum matching  $M$ , *i.e.*  $G$  is a König graph.  $\square$

## 2.3 Stable Matroid Bases

A nonempty family of subsets  $\mathcal{B} \subseteq 2^V$  of a set  $V$  is the family of bases of a *matroid*  $M = (V, \mathcal{B})$  if  $\mathcal{B}$  satisfies the following exchange property for all  $B_1, B_2 \in \mathcal{B}$ :

For each  $x \in B_1 \setminus B_2$ , there exists some  $y \in B_2 \setminus B_1$  such that  $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$ .

Subsets of bases of a matroid  $M$  are said to be *independent* in  $M$ . The dual  $M^* = (V, \mathcal{B}^*)$  of the matroid  $M = (V, \mathcal{B})$  is the matroid with the set of bases

$$\mathcal{B}^* := \{V \setminus B : B \in \mathcal{B}\}.$$

(For more about matroids, see *e.g.* [4]). We are interested in the question whether a matroid  $M$  on the set  $V$  of nodes of a graph  $G = (V, E)$  admits a *stable matroid basis*, *i.e.* a basis of  $M = (V, \mathcal{B})$  that is a stable set in  $G$  as well. This STABLE BASIS PROBLEM is easily seen to  $\mathcal{NP}$ -hard in general, but polynomially solvable for special instances.

**Lemma 2.5.** *The STABLE BASIS PROBLEM is  $\mathcal{NP}$ -complete for  $k$ -uniform matroids.*

*Proof.* The bases of a  $k$ -uniform matroid  $M$  on  $G = (V, E)$  are, by definition, all the subsets of  $V$  with cardinality  $k$ . A stable basis is thus a stable set of size  $k$ , which is  $\mathcal{NP}$ -hard to compute.  $\square$

Polynomially solvable cases of the STABLE BASIS PROBLEM arise, for example, as follows. Let  $G = (V, E)$  be any graph and  $M$  the dual of a partition matroid  $M^*$ . (A *partition matroid* is a matroid whose ground set  $V$  is partitioned into sets  $V = A_1 \cup A_2 \cup \dots \cup A_m$  and the bases are the subsets of  $V$  that contain exactly one element of each of the sets  $A_i$ ).

**Theorem 2.6.** *If the matroid  $M$  is the dual of a partition matroid the STABLE BASIS PROBLEM can be efficiently reduced to an  $R/B$ -SPLIT PROBLEM.*

*Proof.* Construct a graph  $G' = (V, R \cup B)$  with red and blue edges as follows: Let the blue graph  $G_B$  equal  $G$  and the red graph  $G_R$  consist of the disjoint union of  $m$  cliques formed by the sets  $A_1, \dots, A_m$ . By construction,  $V$  can be split into a red stable set  $S_R$  and a blue stable set  $S_B$  if and only if  $V$  can be split into an independent set  $S_R$  of the partition matroid  $M^*$  and a stable set  $S_B$  in the graph  $G$ .

Moreover, as we can always assume the red stable set  $S_R$  to be maximal (with respect to inclusion),  $V$  can be split into a maximal red stable set  $S_R$  and a blue stable set  $S_B$  if and only if  $V$  can be split into a base  $S_R$  of  $M^*$  and a stable set  $S_B$  in  $G$ , *i.e.* if and only if  $S_B$  is a stable basis.  $\square$

The example of Theorem 2.6 shows that the STABLE BASIS PROBLEM becomes polynomial even for general graphs if we restrict the problem to a special class of matroids. The following example exhibits the problem to become polynomial for general matroids  $M$  when we restrict ourselves to the special class of graphs  $G$  where  $G$  is the cocomparability graph of a *tree-order*  $P$ , *i.e.* a partial order  $P$  whose Hasse diagram forms a rooted tree. (Note that a stable set in the cocomparability graph of a partial order is simply a chain in that order.)

**Theorem 2.7.** *If  $P$  is a tree-order, then a maximal independent chain can be calculated in polynomial time.*

*Proof.* In a tree-order, each leaf  $i$  in the Hasse diagram is a maximal element of a unique chain  $C_i$  in  $P$ . For each leaf  $i$  calculate a subchain of  $C_i$  of maximal cardinality that is independent in  $M$ . This can be done easily by calculating a basis of the restricted matroid  $M_i := (C_i, \mathcal{B}_i)$  where  $\mathcal{B}_i = \{|B \cap C_i| : B \in \mathcal{B}\}$  with the matroid greedy-algorithm. If there exists a basis  $B_i$  in  $M_i$  that is a basis in  $M$ , we know that  $B_i$  is a basis in  $M$  that is chain in the partial order  $P$  and therefore a stable basis in the complement graph of the comparability graph of  $P$ .  $\square$

The general problem of a maximal independent chain (or antichain) turns out to be  $\mathcal{NP}$ -complete even for a partition matroid and series-parallel orders. In Section 4 we will prove

**Theorem 2.8.** *The STABLE BASIS PROBLEM is  $\mathcal{NP}$ -complete for a partition matroid and the comparability graph of a series-parallel order.*

### 3 Maximal Covers by Stable Sets

If  $G = (V, R \cup B)$  is not an  $R/B$ -split graph one might ask for the largest subset of  $V$  such that the induced subgraph is an  $R/B$ -split graph. We refer to this problem as the MAX  $R/B$ -SPLIT PROBLEM. The general MAX  $R/B$ -SPLIT PROBLEM is  $\mathcal{NP}$ -hard as it includes the MAX STABLE SET PROBLEM.

Therefore, it would be interesting to identify polynomially solvable cases of the MAX  $R/B$ -SPLIT PROBLEM. A general construction reduces the problem in  $G$  to the STABLE SET PROBLEM in an associated graph  $H$ :

*Example 3.1.* Let  $V'$  be a copy of  $V$  and consider the graph  $H(G_R, G_B) = (V \cup V', E)$  with

$$(i, j) \in E \iff \begin{cases} i, j \in V \text{ and } (i, j) \in R \\ i, j \in V' \text{ and } (i, j) \in B \\ i \in V, j \in V' \text{ and } j \text{ is the copy of } i. \end{cases}$$

This 1-colored graph  $H$  is constructed by joining the red and the blue graph by a (special) perfect matching. A maximal stable set  $S$  of  $H(G_R, G_B)$  corresponds to a maximal union of a red and a blue stable set by setting  $S_R = S \cap V$  and  $S_B = S \cap V'$  resp.  $S = S_R \cup S_B$ . Therefore the MAX  $R/B$ -SPLIT PROBLEM is polynomially solvable if and only if a maximal stable set in  $H(G_R, G_B)$  can be determined in polynomial time.

Example 3.1 shows that the MAX R/B-Split Problem can always be solved by solving a MAX STABLE SET PROBLEM. Would it be also possible to identify pairs of a red and a blue graph where the MAX R/B-SPLIT PROBLEM is always solvable independent of the way the red and the blue edges interact? One example of such a class of tractable pairs is the following:

*Example 3.2.* If  $G_R$  is the complement of a chordal graph  $\bar{G}_R$  (*i.e.* every cycle in  $\bar{G}_R$  of length at least 4 possesses a chord) and  $G_B$  is a comparability graph the MAX R/B-SPLIT PROBLEM is polynomial.

*Proof.* We have to split the nodes into a clique in the chordal graph  $\bar{G}_R$  and an antichain in the partial order corresponding to  $G_B$ . It can be assumed that in a maximal union of a red clique and a blue stable set the red clique is a maximal clique (with respect to inclusion). For the case where  $\bar{G}_R$  is a chordal graph, Fulkerson and Gross [5] showed that all maximal cliques in  $\bar{G}_R$  can be calculated in time  $\mathcal{O}(|V|+|R|)$ . Therefore, it only remains to determine a maximal antichain in the poset  $G_B \setminus C$  for all maximal cliques  $C$  in  $\bar{G}_R$ , which is a polynomial task.  $\square$

Note that the same approach of the preceding proof works for any pair of graphs where the maximal stable sets of the red graph can be listed in polynomial time and a maximal stable set of the blue graph can be calculated efficiently even if the blue graph is reduced by a subset of nodes.

If the red graph and the blue graph are identical, the MAX R/B-SPLIT PROBLEM becomes the problem to determine a maximal union of two stable sets. This problem is still  $\mathcal{NP}$ -hard for general graphs as it is a special instance of the known  $\mathcal{NP}$ -hard MAXIMUM INDUCED SUBGRAPH WITH PROPERTY  $\Pi$  PROBLEM (see [10], p. 381). But again, there are graphs for which this problem is solvable:

*Example 3.3.* If  $G_R$  and  $G_B$  are comparability graphs and  $G_R = G_B$  the MAX R/B-SPLIT PROBLEM is polynomial since there exist efficient algorithms for the maximal union of two antichains relative to the same partial order (see *e.g.* [1]).

The last example raises the question whether the MAX R/B-SPLIT PROBLEM is generally polynomial in case  $G_R$  and  $G_B$  are comparability graphs. We will show in the next section that this problem is already  $\mathcal{NP}$ -hard for comparability graphs of series-parallel orders.

## 4 Hardness Results

We have seen that it is easy to decide whether we can cover all nodes of a graph with two colored stable sets or to find two antichains covering a maximal number of elements relative to one partial order.

Suppose now that we are given a red and a blue partial order on the same ground set  $V$  and let  $l_R$  resp.  $l_B$  denote the length of a longest red resp. blue antichain. In the case  $l_R + l_B < |V|$ , it is obviously impossible to cover all elements with a red and a blue antichain. However, we might still wonder if we can find a red and a blue antichain that cover  $l_R + l_B$  elements.

It turns out that the problem of deciding whether there exist two disjoint differently colored maximal antichains is  $\mathcal{NP}$ -complete already on the class of series-parallel orders. This fact directly implies  $\mathcal{NP}$ -hardness of the MAX R/B-SPLIT PROBLEM.

**Theorem 4.1.** *Given two partial orders  $P_R = (V, \leq_R)$  and  $P_B = (V, \leq_B)$  on the same ground set  $V$ , it is  $\mathcal{NP}$ -hard to decide whether there exist maximal antichains  $A_R$  in  $P_R$  and  $A_B$  in  $P_B$  with  $A_R \cap A_B = \emptyset$ .*

*Proof.* We show  $\mathcal{NP}$ -hardness by a reduction from 3-SAT. Consider a 3-SAT instance with  $k$  clauses on  $n$  variables  $x_i$ ,

$$\bigwedge_{j=1}^k (\ell_1^j \vee \ell_2^j \vee \ell_3^j),$$

i.e.  $\ell_p^j \in \{x_1, \dots, x_n, \neg x_1, \dots, \neg x_n\}$  for  $p \in \{1, 2, 3\}$ . The ground set  $V$  contains all literals and their negations, where appearances of the same literal in different clauses are distinguished:

$$V = \{x_i^j, \neg x_i^j \mid \exists p \in \{1, 2, 3\} : x_i^j = \ell_p^j \vee \neg x_i^j = \ell_p^j\}.$$

In the following, when referring to a literal  $\ell_p^j$ , we mean its incarnation in clause  $j$ , i.e.  $\ell_p^j = (\neg)x_i^j$  for the appropriate  $i$ . The red and blue orders are defined as follows:

$$\begin{aligned} \forall j : \quad & \ell_1^j <_R \ell_2^j <_R \ell_3^j \\ \forall i, j, j' : \quad & x_i^j <_B \neg x_i^{j'} \end{aligned}$$

Fig. 1 shows the Hasse diagram of the reduction for an example formula (without the uncomparable items).

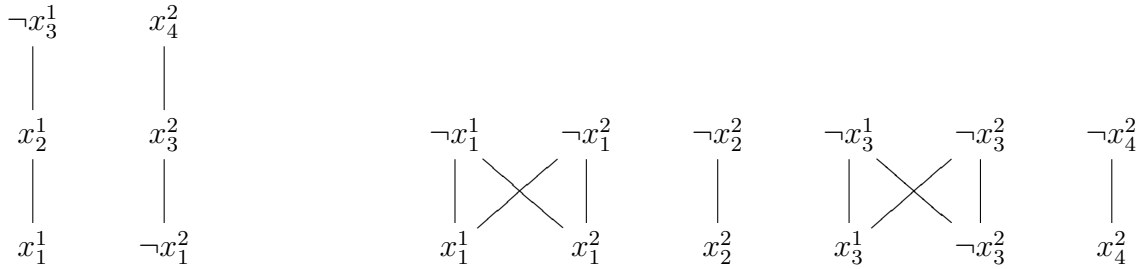


Figure 1: The red order (left) and the blue order (right) for  $(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_3 \vee \neg x_4)$

Obviously, a maximal red antichain covers exactly one literal per clause, whereas a maximal blue antichain corresponds to a consistent assignment of the variables. Note that a maximal red and a maximal blue antichain will be disjoint if and only if the



literals covered by the red antichain are *false* in the variable assignment corresponding to the blue antichain. So if we can find two maximal disjoint antichains, negating the variable assignment corresponding to the blue antichain produces a satisfying variable assignment for the original 3-SAT instance. On the other hand, if there are no such two antichains, there also is no variable assignment satisfying all clauses. As this reduction is obviously polynomial, we have shown  $\mathcal{NP}$ -completeness.  $\square$

*Remark 4.2.* As the orders produced in the reduction of Theorem 4.1 are series-parallel, the MAX R/B-SPLIT PROBLEM is already  $\mathcal{NP}$ -hard for series-parallel orders. Furthermore, finding two maximal disjoint chains or a maximal chain and a disjoint maximal antichain is also  $\mathcal{NP}$ -hard (already in series-parallel orders), as one can demonstrate *via* complementary constructions. This holds because the co-graph of a series-parallel comparability graph is again the comparability graph of a series-parallel order.

The same construction also produces a reduction from MAX 3-SAT. This directly gives some insight into the approximability of the MAX R/B-SPLIT PROBLEM. (For the basic notions of approximation theory, the reader is referred to, *e.g.*, [10]).

**Corollary 4.3.** *For  $\varepsilon > 0$ , there cannot exist a  $(\frac{31}{32} + \varepsilon)$ -approximative algorithm for MAX R/B-SPLIT, unless  $\mathcal{P} = \mathcal{NP}$ .*

*Proof.* Note that it does not make any difference if an element is covered by the red, blue or both antichains. Hence we may assume without loss of generality that the blue antichain is of maximal cardinality  $3k$  and thus corresponds to a consistent assignment of all variables. Then the size of the red antichain is exactly the number of clauses satisfied by the negated variable assignment. As it is  $\mathcal{NP}$ -hard to approximate MAX 3-SAT better than  $7/8$  (see [9]), it is easy to calculate that approximating MAX R/B-SPLIT better than  $31/32$  is also  $\mathcal{NP}$ -hard.  $\square$

We finally remark that taking a maximal red and a maximal blue antichain yields a simple 2-approximation algorithm, which altogether places MAX R/B-SPLIT into the class of so-called APX-complete problems [10].

A slight amendment in the proof of Thm. 4.1 now allows us to proof Thm. 2.8:

*Proof of Thm. 2.8.* Given a 3-SAT formula, this time we need to take only the literals as they appear in the clauses (and not their negations):

$$V = \{x_i^j \mid \exists p \in \{1, 2, 3\} : x_i^j = \ell_p^j\} \cup \{\neg x_i^j \mid \exists p \in \{1, 2, 3\} : \neg x_i^j = \ell_p^j\}.$$

The central observation is now that the red order in the proof of Thm. 4.1 can be substituted by the partition matroid on  $V$  where  $V$  is partitioned into the  $k$  clauses of the 3-SAT formula:

$$V = \bigcup_{j=1}^k \{\ell_1^j, \ell_2^j, \ell_3^j\}$$

The definition of the (blue) order remains the same, but now on only half as many elements. A stable basis here corresponds to a consistent variable assignment satisfying (at least) one literal in each clause.  $\square$

**Corollary 4.4.** *For  $\varepsilon > 0$ , there cannot exist a  $(\frac{7}{8} + \varepsilon)$ -approximative algorithm for finding a longest independent (anti-)chain, unless  $\mathcal{P} = \mathcal{NP}$ .*

*Proof.* In the construction above, the associated variable assignment of an independent antichain of length  $l$  satisfies (at least)  $l$  clauses.  $\square$

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